Cubature Kalman Filters

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Outline of The Lecture

1. Introductory Remarks
2. The Bayesian Filter: A powerful tool for solving the nonlinear filtering problem
3. The Cubature Kalman Filter
4. Example Application: Tracking a manoeuvring ship
5. Concluding Remarks

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1. Introductory Remarks

“Optimality versus Robustness”

In many applications, global optimality may not be practically feasible:

- Large-scale nature of the problem
- Infeasible computability
- Curse-of-dimensionality

Hence, the practical requirement of having to settle for a sub-optimal solution of the system design

- Trade-off global optimality for computational tractability and robust behaviour.
Criterion for sub-optimality

DO AS BEST AS YOU CAN, AND NOT MORE

• This statement is the essence of what the human brain does on a daily basis:

    Provide the “best” solution in the most reliable fashion for the task at hand, given limited resources.

• Key question: How do we define “best”? 
2. The Bayesian Filter: A powerful tool for solving the nonlinear tracking problem

Problem statement:

Given a nonlinear dynamic system, estimate the hidden state of the system in a recursive manner by processing a sequence of noisy observations dependent on the state.

• The Bayesian filter provides a unifying framework for the optimal solution of this problem, at least in a conceptual sense.

Unfortunately, except in a few special cases, the Bayesian filter is not implementable in practice -- hence the need for approximation.
Bayesian Filter (continued)

State-space Model

1. **System (state) Model**
   \[ x_{t+1} = a(x_t) + \omega_t \]

2. **Measurement model**
   \[ y_t = b(x_t) + \nu_t \]

where \( t \) = discrete time
   \[ x_t = \text{state at time } t \]
   \[ y_t = \text{observation at time } t \]
   \[ \omega_t = \text{dynamic noise} \]
   \[ \nu_t = \text{measurement noise} \]
Bayesian Filter (continued)

Assumptions:

- Nonlinear functions $a(\cdot)$ and $b(\cdot)$ are known

- Dynamic noise $\omega_t$ and measurement noise $\nu_t$ are statistically independent Gaussian processes of zero mean and known covariance matrices.
Bayesian filter (continued)

1. Time-update equation:

\[ p(x_t | Y_{t-1}) = \int_{\mathbb{R}^n} p(x_t | x_{t-1}) p(x_{t-1} | Y_{t-1}) dx_{t-1} \]

where \( \mathbb{R}^n \) denotes the \( n \)-dimensional state space.

2. Measurement-update equation:

\[ p(x_t | Y_t) = \frac{1}{Z_t} p(x_t | Y_{t-1})l(y_t | x_t) \]

where \( Z_t \) is the normalizing constant defined by

\[ Z_t = \int_{\mathbb{R}^n} p(x_t | Y_{t-1})l(y_t | x_t) dx_t \]
Bayesian Filter (continued)

• The celebrated Kalman filter is a special case of the Bayesian filter, assuming that the dynamic system is linear and both the dynamic noise and measurement noise are statistically independent processes.

• Except for this special case and couple of other cases, exact computation of the predictive distribution $p(x_t|Y_{t-1})$ is not feasible.

• We therefore have to abandon optimality and be content with a sub-optimal nonlinear filtering algorithm that is computationally tractable.
Bayesian Filtering (continued)

Two Approaches for Approximate Nonlinear Filtering

1. **Direct numerical approximation of the posterior in a local sense:**
   - Extended Kalman filter (simple and therefore widely used)
   - Unscented Kalman filter (heuristic in its formulation)
   - Central-difference Kalman filter
   - Cubature Kalman filter (New)

2. **Indirect numerical approximation of the posterior in a global sense:**
   - Particle filters:
     - Roots embedded in Monte Carlo simulation
     - Computationally demanding
3. The Cubature Kalman Filter


- At the heart of the Bayesian filter, we have to compute integrals whose integrands are expressed in the common form

  \[(\text{Nonlinear function}) \times (\text{Gaussian function})\]

- The challenge is to numerically approximate the integral so as to completely preserve second-order information about the state \(x_t\) that is contained in the sequence of observations \(y_t\)

- The computational tool that accommodates this requirement is the cubature rule.
Cubature Kalman Filter (continued)

The Cubature Rule

- In mathematical terms, we have to compute an integral of the generic form

\[ h(f) = \int_{\mathbb{R}^n} f(x) \exp\left(-\frac{1}{2} x^T x\right) dx \]  

(1)

- To do the computation, a key step is to make a change of variables from the Cartesian coordinate system (in which the vector \(x\) is defined) to a spherical-radial coordinate system:

\[ x = rz \text{ subject to } z^T z = 1 \text{ and } x^T x = r^2 \text{ where } 0 \leq r < \infty \]

- The next step is to apply the radial rule using the Gaussian quadrature.
Properties of the Cubature Kalman Filter

Property 1: The cubature Kalman filter (CKF) is a *derivative-free on-line sequential-state estimator*: It relies on integration for its operation.

Property 2: Approximations of the moment integrals are all *linear* in the number of function evaluations.

Property 3: *Computational complexity* of the cubature Kalman filter as a whole, grows as $n^3$, where $n$ is the dimensionality of the state space.

Property 4: The cubature Kalman filter completely *preserves second-order information about the state* that is contained in the observations.

Property 5: The cubature Kalman filter *inherits properties of the linear Kalman filter*, including square-root filtering for improved accuracy and reliability.

Property 6: The cubature Kalman filter is the closest known direct approximation to the Bayesian filter, outperforming the extended Kalman filter and the central-difference Kalman filter:

*It eases the curse-of-dimensionality problem but, by itself, does not overcome it.*
4. Example Application: Tracking a Manoeuvring Ship

Problem statement:

Track a ship moving in an area bounded by a shore line, assumed to be a circular disc of known radius and centered at the origin.

- The ship’s motion is modelled by a constant velocity perturbed by additive white Gaussian noise.

- When the ship tries to drift outside the shoreline, a gentle turning force pushes it back towards the origin.

- The model is interesting in that it exhibits a nonlinear behavior near the shoreline, thereby providing a good test for assessing the performance of different nonlinear filters.
Tracking a Manoeuvring Ship

- Dynamic State-space Model:

\[
\dot{x}_t = [\dot{\xi}, \dot{\eta}, f_1(x_t), f_2(x_t)]^T + \sqrt{Q_t} \beta_t
\]  \hspace{1cm} (1)

\[
\begin{pmatrix}
  r_k \\
  \theta_k
\end{pmatrix}
= \begin{pmatrix}
  \sqrt{\xi_k^2 + \eta_k^2} \\
  \tan^{-1}(\frac{\eta_k}{\xi_k})
\end{pmatrix}
+ w_k
\]  \hspace{1cm} (2)

- where

\[
f_1(x) = \begin{cases}
  -K\xi \\
  \frac{\sqrt{\xi^2 + \eta^2}}{\sqrt{\xi^2 + \eta^2}}, \\
  0,
\end{cases}
\quad \text{if } \sqrt{\xi^2 + \eta^2} \geq r \text{ and } \xi \dot{\xi} + \eta \dot{\eta} \geq 0;
\]

\[
f_2(x) = \begin{cases}
  -K\eta \\
  \frac{\sqrt{\xi^2 + \eta^2}}{\sqrt{\xi^2 + \eta^2}}, \\
  0,
\end{cases}
\quad \text{if } \sqrt{\xi^2 + \eta^2} \geq r \text{ and } \xi \dot{\xi} + \eta \dot{\eta} \geq 0;
\]
Tracking (continued)

- Used Euler method with 5 steps b/w each measurement interval to numerically integrate (1)

- Data:
  - Radius of the disk-shape shore, \( r = 5 \) units
  - Gaussian process noise intensity, \( Q = 0.01 \)
  - Gaussian measurement noise parameters, \( \sigma_r = 0.01 \) and \( \sigma_\theta = \frac{0.5\pi}{180} \)
  - Estimated initial state, \( \hat{x}_{0|0} = [1111]^T \) and covariance, \( P_{0|0} = 10I_4 \)
  - Radar scans = 1000/Monte Carlo run
  - 50 independent Monte Carlo runs
Motion of the ship.

Figure 1: I - initial point, F - final point, ★ - Radar location
Performance Comparison

Figure 2: dashed red- PF (1000 particles), thin blue- CDKF, dark black- CKF

RMSE in position
Performance Comparison (continued)

Figure 3: dashed red- PF (1000 particles), thin blue- CDKF, dark black- CKF
5. Concluding Remarks

The results presented on the tracking of a manoeuvring ship, with constraints imposed on its motion, demonstrate the following:

1. Both the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) fail when tested in a highly nonlinear environment.

2. The cubature Kalman filter (CKF) outperforms the central difference Kalman filter (CDKF) and particle filter (PF).
Simply stated:

• The cubature Kalman filter and its square-root extension provide a new set of powerful tools for solving nonlinear state-estimation tracking problems.

• Cubature Kalman filters provide the closest approximation to a Bayesian filter, which is optimal (the best we can do), at least in a conceptual sense.
Reference