Cubature Kalman Filters

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Outline of The Lecture

1. **Introductory Remarks**
2. **The Bayesian Filter: A powerful tool for solving the nonlinear filtering problem**
3. **The Cubature Kalman Filter**
4. **Example Application: Tracking a manoeuvring ship**
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Reference

1. Introductory Remarks

"Optimality versus Robustness"

In many applications, global optimality may not be practically feasible:

- **Large-scale nature of the problem**
- **Infeasible computability**
- **Curse-of-dimensionality**

Hence, the practical requirement of having to settle for a sub-optimal solution of the system design

- **Trade-off global optimality for computational tractability and robust behaviour.**

Criterion for sub-optimality

DO AS BEST AS YOU CAN, AND NOT MORE

- **This statement is the essence of what the human brain does on a daily basis:**

  Provide the “best” solution in the most reliable fashion for the task at hand, given limited resources.

- **Key question: How do we define “best”?**

2. The Bayesian Filter: A powerful tool for solving the nonlinear tracking problem

Problem statement:

Given a nonlinear dynamic system, estimate the *hidden state* of the system in a recursive manner by processing a sequence of *noisy observations* dependent on the state.

- **The Bayesian filter provides a unifying framework for the optimal solution of this problem, at least in a conceptual sense.**

  Unfortunately, except in a few special cases, the Bayesian filter is not implementable in practice -- hence the need for approximation.

Bayesian Filter (continued)

State-space Model

1. **System (state) Model**

$$\mathbf{x}_{t+1} = \mathbf{a}(\mathbf{x}_t) + \omega_t$$

2. **Measurement model**

$$\mathbf{y}_t = \mathbf{b}(\mathbf{x}_t) + \nu_t$$

where t **= discrete time**

$\mathbf{x}_t$ **= state at time** t

$\mathbf{y}_t$ **= observation at time** t

ω_t **= dynamic noise**

ν_t **= measurement noise**

Bayesian Filter (continued)

Assumptions:

- **Nonlinear functions a(·) and b(·) are known**

- **Dynamic noise ω_t and measurement noise ν_t are statistically independent Gaussian processes of zero mean and known covariance matrices.**

Bayesian filter (continued)

1. Time-update equation:

$$\underbrace{p(\mathbf{x}_t|\mathbf{Y}_{t-1})}_{\text{Predictive distribution}} = \int_{R^n} \underbrace{p(\mathbf{x}_t|\mathbf{x}_{t-1})}_{\text{Prior distribution}} \underbrace{p(\mathbf{x}_{t-1}|\mathbf{Y}_{t-1})}_{\text{Old posterior distribution}} d\mathbf{x}_{t-1}$$

where R^n denotes the n-dimensional state space.

2. Measurement-update equation:

$$\underbrace{p(\mathbf{x}_t|\mathbf{Y}_t)}_{\text{Updated posterior distribution}} = \frac{1}{Z_t} \underbrace{p(\mathbf{x}_t|\mathbf{Y}_{t-1})}_{\text{Predictive distribution}} \underbrace{l(\mathbf{y}_t|\mathbf{x}_t)}_{\text{Likelihood function}}$$

where Z_t is the normalizing constant defined by

$$Z_t = \int_{R^n} p(\mathbf{x}_t|\mathbf{Y}_{t-1}) l(\mathbf{y}_t|\mathbf{x}_t) d\mathbf{x}_t$$

Bayesian Filter (continued)

- **The celebrated Kalman filter is a special case of the Bayesian filter, assuming that the dynamic system is linear and both the dynamic noise and measurement noise are statistically independent processes.**

- **Except for this special case and couple of other cases, exact computation of the predictive distribution $p(\mathbf{x}_t | \mathbf{Y}_{t\text{-}1})$ is not feasible.**

- **We therefore have to abandon optimality and be content with a sub-optimal nonlinear filtering algorithm that is computationally tractable.**

Bayesian Filtering (continued)

Two Approaches for Approximate Nonlinear Filtering

1. **Direct numerical approximation of the posterior in a local sense:**
   - **Extended Kalman filter (simple and therefore widely used)**
   - **Unscented Kalman filter (heuristic in its formulation)**
   - **Central-difference Kalman filter**
   - **Cubature Kalman filter (New)**

2. **Indirect numerical approximation of the posterior in a global sense:**
   - **Particle filters:**
     Roots embedded in Monte Carlo simulation
     Computationally demanding

3. The Cubature Kalman Filter

(Arasaratnam and Haykin, IEEE Trans. Automatic Control, to appear in 2009, June.

- **At the heart of the Bayesian filter, we have to compute integrals whose integrands are expressed in the common form**

(Nonlinear function) x (Gaussian function)

- **The challenge is to numerically approximate the integral so as to completely preserve second-order information about the state $\mathbf{x}_t$ that is contained in the sequence of observations $\mathbf{y}_t$**

- **The computational tool that accommodates this requirement is the *cubature rule.***

Cubature Kalman Filter (continued)

The Cubature Rule

- **In mathematical terms, we have to compute an integral of the generic form**

$$h(\mathbf{f}) = \int_{R^n} \underbrace{\mathbf{f}(\mathbf{x})}_{\text{Arbitray nonlinear function}} \underbrace{\exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{x}\right)}_{\text{Normalized Gaussian function of zero mean and unit covariance matrix}} d\mathbf{x} \quad (1)$$

- **To do the computation, a key step is to make a change of variables from the Cartesian coordinate system (in which the vector x is defined) to a spherical-radial coordinate system:**

$$\mathbf{x} = r\mathbf{z} \text{ subject to } \mathbf{z}^T\mathbf{z} = 1 \text{ and } \mathbf{x}^T\mathbf{x} = r^2 \text{ where } 0 \le r < \infty$$

- **The next step is to apply the radial rule using the Gaussian quadrature.**

Properties of the Cubature Kalman Filter

***Property* 1: The cubature Kalman filter (CKF) is a *derivative-free on-line sequential-state estimator*: It relies on integration for its operation.**

***Property* 2: Approximations of the moment integrals are all *linear* in the number of function evaluations.**

***Property* 3: Computational complexity of the cubature Kalman filter as a whole, grows as n^3, where n is the dimensionality of the state space.**

***Property* 4: The cubature Kalman filter *completely preserves second-order information* about the state that is contained in the observations.**

***Property* 5: The cubature Kalman filter *inherits properties of the linear Kalman filter*, including square-root filtering for improved accuracy and reliability.**

***Property* 6: The cubature Kalman filter is the closest known direct approximation to the Bayesian filter, outperforming the extended Kalman filter and the central-difference Kalman filter:**

It eases the curse-of-dimensionality problem
but, by itself, does not overcome it.

4. Example Application: Tracking a Manoeuvring Ship

Problem statement:

> **Track a ship moving in an area bounded by a shore line, assumed to be a circular disc of known radius and centered at the origin.**

- **The ship's motion is modelled by a constant velocity perturbed by additive white Gaussian noise.**
- **When the ship tries to drift outside the shoreline, a gentle turning force pushes it back towards the origin.**
- **The model is interesting in that it exhibits a nonlinear behavior near the shoreline, thereby providing a good test for assessing the performance of different nonlinear filters.**

Tracking a Manoeuvring Ship

- **Dynamic State-space Model:**

$$\dot{\mathbf{x}}_t = [\dot{\xi}_t \dot{\eta} f_1(\mathbf{x}_t) f_2(\mathbf{x}_t)]^T + \sqrt{\mathbf{Q}_t}\beta_t \qquad \textbf{(1)}$$

$$\begin{pmatrix} r_k \\ \theta_k \end{pmatrix} = \begin{pmatrix} \sqrt{\xi_k^2 + \eta_k^2} \\ \tan^{-1}\left(\frac{\eta_k}{\xi_k}\right) \end{pmatrix} + \mathbf{w}_k \qquad \textbf{(2)}$$

- **where**

$$f_1(\mathbf{x}) = \begin{cases} \frac{-K\xi}{\sqrt{\xi^2 + \eta^2}}, & \sqrt{\xi^2 + \eta^2} \geq r \text{ and } \xi\dot{\xi} + \eta\dot{\eta} \geq 0; \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(\mathbf{x}) = \begin{cases} \frac{-K\eta}{\sqrt{\xi^2 + \eta^2}}, & \sqrt{\xi^2 + \eta^2} \geq r \text{ and } \xi\dot{\xi} + \eta\dot{\eta} \geq 0; \\ 0, & \text{otherwise} \end{cases}$$

Tracking (continued)

- **Used Euler method with 5 steps b/w each measurement interval to numerically integrate (1)**

- **Data:**

  - **Radius of the disk-shape shore, *r* = 5 units**
  - **Gaussian process noise intensity, *Q* = 0.01**
  - **Gaussian measurement noise parameters, σ_r = 0.01 and** $\sigma_\theta = \frac{0.5\pi}{180}$
  - **Estimated initial state,** $\hat{\mathbf{x}}_{0|0} = [1111]^T$ **and covariance,** $P_{0|0} = 10\mathbf{I}_4$
  - **Radar scans = 1000/Monte Carlo run**
  - **50 independent Monte Carlo runs**

Motion of the ship.

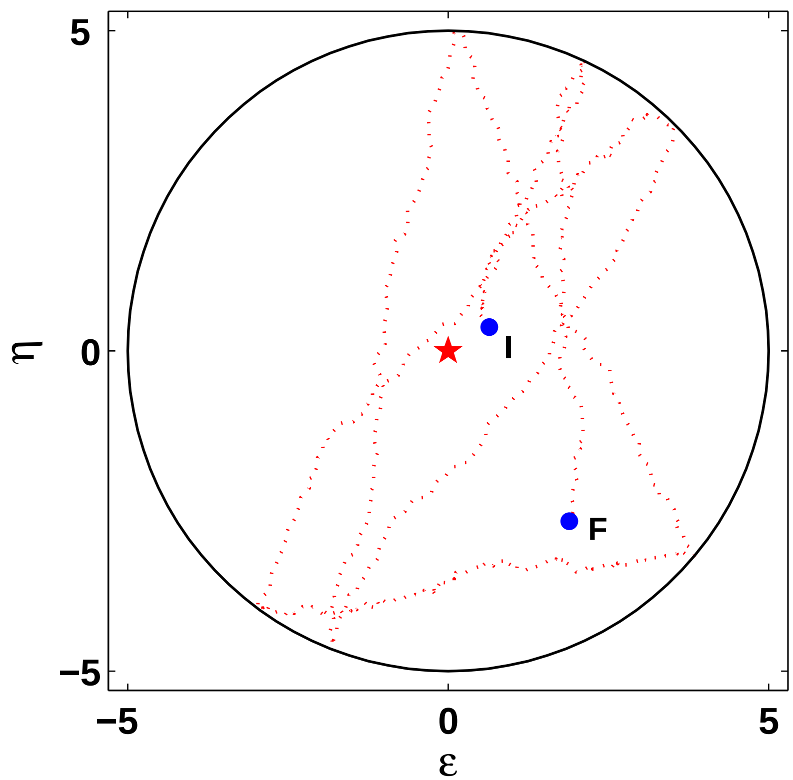
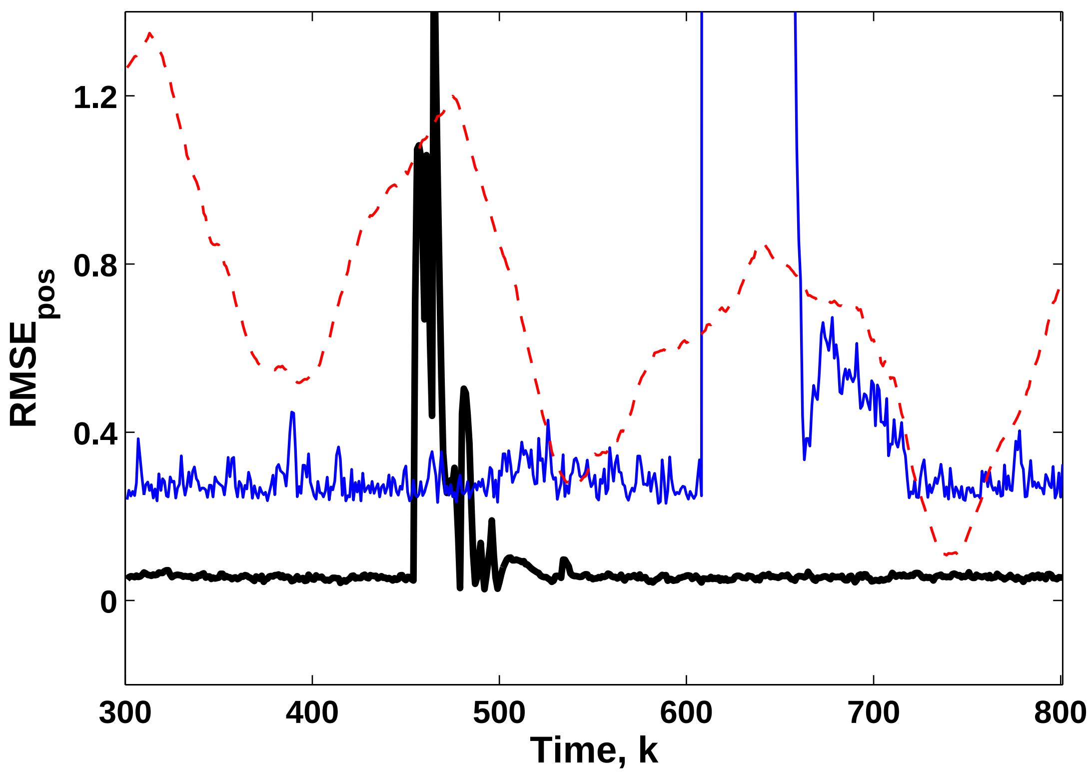


Figure 1: I - initial point, F - final point, ★ - Radar location

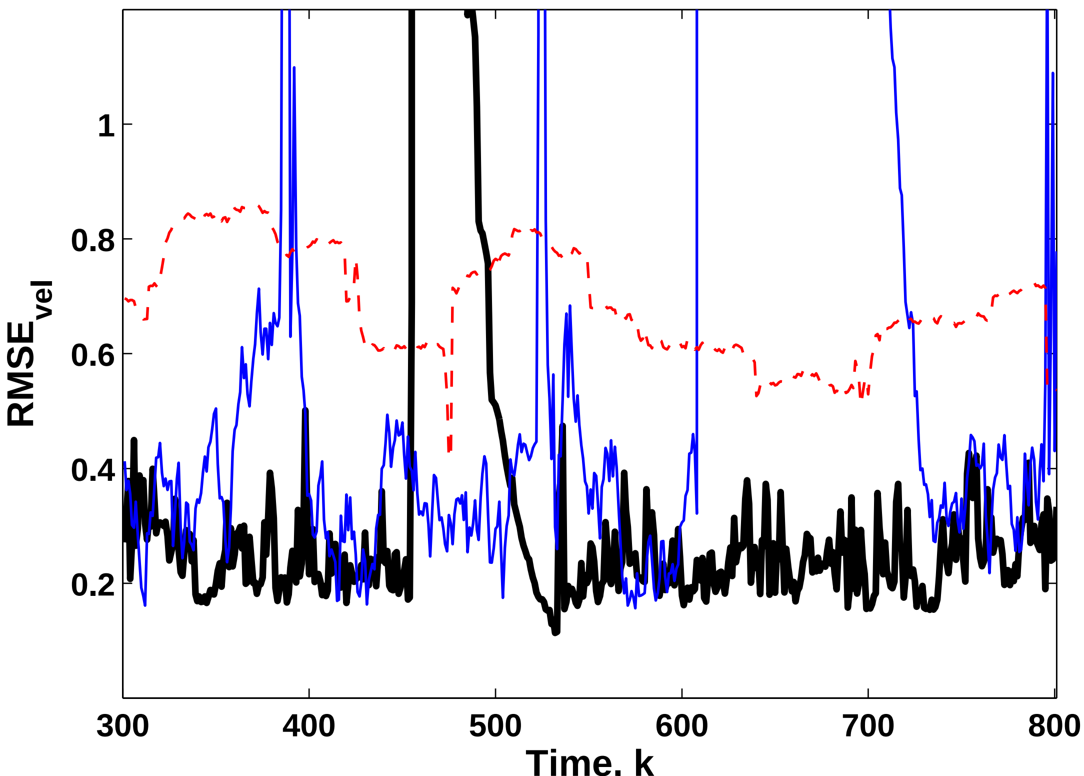
Performance Comparison



RMSE in position

Figure 2: dashed red- PF (1000 particles), thin blue- CDKF, dark black- CKF

Performance Comparison (continued)



RMSE in velocity

Figure 3: dashed red- PF (1000 particles), thin blue- CDKF, dark black- CKF

5. Concluding Remarks

The results presented on the tracking of a manoeuvring ship, with constraints imposed on its motion, demonstrate the following:

1. **Both the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) fail when tested in a highly nonlinear environment.**

2. **The cubature Kalman filter (CKF) outperforms the central difference Kalman filter (CDKF) and particle filter (PF).**

Simply stated:

- **The cubature Kalman filter and its square-root extension provide a new set of powerful tools for solving nonlinear state-estimation tracking problems.**

- **Cubature Kalman filters provide the closest approximation to a Bayesian filter, which is optimal (the best we can do), at least in a conceptual sense.**

Reference

I. Arasaratnam and s. Haykin, "Cubature Kalman Filters", IEEE Trans. Automatic Control, 2009, June.